TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 271, Number 1, May 1982

TYPE STRUCTURE COMPLEXITY AND DECIDABILITY[1]

BY

T. S. MILLAR

ABSTRACT. We prove that for every countable homogeneous model $\mathcal{Q}$ such that the set of recursive types of $\mathrm{Th}(\mathcal{Q})$ is Σ_2^0, $\mathcal{Q}$ is decidable iff the set of types realized in $\mathcal{Q}$ is a Σ_2^0 set of recursive types. As a corollary to a lemma, we show that if a complete theory T has a recursively saturated model that is decidable in the degree of T, then T has a prime model.

In this paper all models mentioned will be assumed countable. If $\mathcal{Q}$ is homogeneous and realizes either no nonprincipal types **[1]** or all recursive types **[2]** then

(*) $\mathcal{Q}$ is decidable iff the set of types realized by $\mathcal{Q}$ is an r.e. set of recursive types.

An examination of the proofs involved leads naturally to the conjecture that the techniques can be combined to prove (*) for those $\mathcal{Q}$ that are simply homogeneous. Unfortunately this is false in general **[5]**. However, if the structure of recursive types of the theory of $\mathcal{Q}$ is not pathological, then (*) can be proved for those $\mathcal{Q}$ which are simply homogeneous. Specifically, the principal result of this paper is to prove that if $\mathcal{Q}$ is homogeneous and the set of recursive types of the theory of $\mathcal{Q}$ is Σ_2^0, then $\mathcal{Q}$ is decidable iff the set of types realized by $\mathcal{Q}$ is a Σ_2^0 set of recursive types. Since every complete theory's set of recursive types is Π_2^0, the result is the best possible, in light of **[5]**.

Notations and conventions. All types in this paper are assumed complete. A specific effective first order language L is assumed fixed, as well as an effective enumeration $\{\sigma_i \mid i < \omega\}$ of all formulas of the language. An n-type Γ is recursive if the set $\{i \mid \sigma_i \in \Gamma(x_0, \ldots, x_{n-1})\}$ is recursive. $\{\mu_i \mid i < \omega\}$ is an effective enumeration of all partial recursive functions $\mu\colon \omega \to 2$. An index e for a recursive type Γ is a natural number such that μ_e is the characteristic function for Γ relative to $\{\sigma_i \mid i < \omega\}$. A set of recursive types is Σ_1^0 (Σ_2^0) if there is a Σ_1^0 (Σ_2^0) set of indices for the types in that set. We will say $\{\Gamma_i \mid i < \omega\}$ is an effective enumeration of types if there is some recursive f such that $f(i)$ is an index for Γ_i, $i < \omega$. Γ^s will denote the first s formulas (order determined by index) of $\Gamma \cap \{\sigma_i \mid i < \omega\}$. $\theta^k = \theta$ if $k = 0$, and $\neg\theta$ if $k = 1$. $\{c_i \mid i < \omega\}$ will be distinct constant symbols not in L and $\{\psi_i \mid i < \omega\}$ an effective enumeration of all sentences in $L \cup \{c_i \mid i < \omega\}$ such that each sentence occurs

Received by the editors March 14, 1980.

1980 *Mathematics Subject Classification*. Primary 03F50; Secondary 03H13.

[1]The preparation of this paper was partially supported by Grant NSF-MCS-7900824.

0002-9947/81/0000-1054/$03.00

infinitely often. $\{\bar{c}_i \mid i < \omega\}$ will be an effective enumeration of $\{c_i \mid i < \omega\}^{<\omega}$. If $\theta(c_0, \ldots, c_n) \in L \cup \{c_i \mid i < \omega\}$ such that all c_j's occurring in θ are among $\{c_i \mid i \leqslant n\}$, then

$$\theta^*\left[\left(c_{i_0}, \ldots, c_{i_s}\right)\right] =_{\mathrm{df}} \exists x_0 \cdots \exists x_{i_0-1} \exists x_{i_0+1} \cdots \exists x_{i_s-1} \exists x_{i_s+1} \cdots \exists x_n \theta\left(x_0, \ldots, x_{i_0-1}, c_{i_0}, x_{i_0+1}, \ldots, x_{i_s-1}, c_{i_s}, x_{i_s+1}, \ldots, x_n\right).$$

v: $\omega \times \omega \to \omega$ is a recursive function that is 1-1 onto and such that if $v(i,j) = n$ then $i \leqslant n$. $(r)_i$ is the exponent of the ith prime in the prime factorization of r. Finally, if F is a function with domain ω, then 'F^*' denotes '$\mathrm{Lim}_{n\to\infty} F(n)$'. For other definitions and conventions, see [**2**–**4**].

We begin with several lemmas and interesting corollaries.

LEMMA 1. *If the set of recursive types of a complete theory is Σ_2^0, then it is also Σ_1^0.*

PROOF. This is immediate from Lemma 3 of [**2**].

By the familiar technique of padding, if B is a set of recursive types with an r.e. set of indices, then it is easy to see that it has a recursive set A of indices. However, to then say that B is recursive would be misleading, since the question of whether or not n is an index of a type in B is not equivalent to whether or not n is in A. However, the use of the term r.e. is not similarly misleading.

LEMMA 2. *Assume that the set of all recursive types of a complete theory T is Σ_2^0. Then for every Σ_2^0 set A of types of T there is a Σ_1^0 set of types $B \supset A$ such that every type in $B - A$ is principal.*

PROOF. Assume for notational simplicity that A is a set of 1-types. Let $\{\varphi_i(x) \mid i < \omega\}$ be an effective enumeration of all formulas of $L(T)$ in the one free variable displayed, and let $\{\Sigma_i \mid i < \omega\}$ be an effective enumeration (by Lemma 1) of all recursive 1-types of T. Since A is Σ_2^0, fix a recursive $R(x, y, n)$ such that $\{n \mid \exists x \forall y R(x, y, n)\}$ is a set of indices for the types in A. We will define $f(n, m)$, inductively on m and uniformly in n, such that for the recursive h produced by the *s-m-n* theorem satisfying $f(n, m) = \mu_{h(n)}(m)$ for all $m, n, < \omega$, the range of h will be the r.e. set of indices (relative to $\{\varphi_i \mid i < \omega\}$), for the desired B. So fix $n < \omega$. We specify that n is *active* for $m = 0$. Assume that $f(n, m')$ has been defined for $m' < m$. If n is active for m then define f according to:

(a) $f(n, m) = \mu_{(n)_1}(m)$ if

$$\exists s \geqslant m\left[\mu^s_{(n)_1}(m)\downarrow \text{ and } \forall r \leqslant s\, R\left((n)_2, r_1(n)_1\right)\right]$$

and

$$T \vdash \exists x\left[\bigwedge_{i<m} \varphi_i^{f(n,i)}(x) \wedge \varphi_m^{\mu_{(n)_1}(m)}\right];$$

(b) $f(n, m) = 0$ if not (a) and

$$T \vdash \exists x\left[\bigwedge_{i<m} \varphi_i^{f(n,i)}(x) \wedge \varphi_m(x)\right];$$

(c) $f(n, m) = 1$ otherwise.

If n is not active at m then:

(i) $f(n, m) = k$ if

$$T \vdash \left[\bigwedge_{i<m} \varphi_i^{f(n,i)}(x) \to \varphi_m(x)^k \right], \qquad k = 0, 1;$$

(ii) $f(n, m) = 1 - k$ otherwise, where for the least j such that

$$\bigwedge_{i<m} \varphi_i^{f(n,i)} \in \Sigma_j, \qquad \varphi_m^k \in \Sigma_j, \qquad k = 0, 1.$$

If the defining condition is (a) then n is active at $m + 1$, otherwise n is inactive.

First we claim that f is recursive. The only condition that is not immediate is when n is active at m, since there is an unbounded quantifier in the defining formula of (a). However, if $\mu_{(n)_1}(m)\uparrow$, then certainly $(n)_1$ is not the index of a type. Therefore $\exists r \neg R((n)_2, r, (n)_1)$, and so the search in (a) would terminate. Next we claim that every type in A has an index in the range of h. For suppose $\exists x \forall y$ $R(x, y, v)$. Then for an r such that $\forall y R(r, y, v)$ it is easy to see that $h(3^r \cdot 2^v)$ is an index for the type in A with index v. Finally, assume that a nonprincipal recursive type $\Sigma_j \notin A$ has an index $h(n)$, in order to obtain a contradiction. Fix the least subscripted such Σ_{j_0}. Since $\Sigma_{j_0} \notin A$, fix a y such that $\neg R((n)_2, y, (n)_1)$. So by condition (a), n is not active for $m > y$. By the choice of j_0 there is an $s_0 > y$ such that $\bigwedge_{i<s_0} \varphi_i^{f(n,i)} \notin \Sigma_j$, $j < j_0$. Since Σ_{j_0} is nonprincipal, there is a φ_p, $p > s_0$, such that

$$T \nvdash \left[\bigwedge_{i<s_0} \varphi_i^{f(n,i)}(x) \to \varphi_p^k \right], \qquad k = 0, 1.$$

Then for some $m \leqslant p$ we must have $\bigwedge_{i\leqslant m} \varphi_i^{f(n,i)} \notin \Sigma_{j_0}$, the desired contradiction. This completes the lemma.

Corollary 1. *If the set of recursive types of a complete decidable theory T is Σ_2^0, then T has a prime model.*

Proof. First we take A to be empty in the previous lemma. However, we modify the subsequent proof so that for φ_n consistent with T, $f(n, j)$ is defined so that

$$T \vdash \exists x \left[\bigwedge_{i<n} \varphi_i^{f(n,i)}(x) \wedge \varphi_n(x) \right], \qquad f(n, n) = 0.$$

Then $f(n, m)$ is defined as in the construction for $m > n$. Thus $h(n)$ will be the index of a principal type containing φ_n, for those φ_n consistent with T. By well-known results this implies that T has a prime model. In fact, by **[1]** the prime model of T is then decidable. Note that this corollary is applicable to arbitrary complete theories, except of course everything must be relativized to the degree of the theory involved.

Corollary 2. *If a complete theory has a recursively saturated model which is decidable in the degree of the theory, then the theory has a prime model.*

Proof. Every type of a complete theory T which is recursive in the degree of T is realized in every recursively saturated model of T. Also, if a model $\mathfrak{A}$ is decidable in some degree **a**, then the set of types realized in $\mathcal{A}$ is r.e. in the degree **a**. Now just apply the relativized version of the previous corollary. Again by [**1**], the prime model is actually decidable in the degree of the theory.

THEOREM. *Assume that the set of recursive types of* $\mathrm{Th}(\mathcal{A})$ *is* Σ^0_2 *and* $\mathcal{A}$ *is homogeneous. Then* $\mathcal{A}$ *is decidable iff the set of types realized in* $\mathcal{A}$ *is a* Σ^0_2 *set of recursive types.*

PROOF. The 'only if' is immediate. For the other direction fix, by Lemmas 1 and 2, effective enumerations $\{\Sigma_i \mid i < \omega\}$ and $\{\Gamma_i \mid i < \omega\}$ of all recursive types of $\mathrm{Th}(\mathcal{A})$ and of those types realized in $\mathcal{A}$, respectively. It is sufficient to construct a decidable homogeneous model $\mathcal{C}$ realizing exactly the set of types $\{\Gamma_i \mid i < \omega\}$. In fact, we will only construct the complete diagram of such a model. This will be done by a Henkin construction that a stage t inductively determines a $\theta_t \in \{\psi_i \mid i < \omega\}$. The complete diagram will be $\{\theta_i \mid i < \omega\}$. We adopt the abbreviation $\chi_j =_{\mathrm{df}} \bigwedge_{i<j} \theta_i$. Partial recursive functions f_i, $g_{i,j}$: $\omega \to \{\bar{c}_i \mid i < \omega\}$, H_i: $\omega \to \{\Gamma_i \mid i < \omega\}$ will also be inductively defined during the course of the construction. In terms of these functions we define the partial recursive A_i: $\omega \to \{\bar{c}_j \mid j < \omega\}$, $i < \omega$ by $A_0(s) = \langle\ \rangle$, and $A_{5t+i}(s)$ is to be the smallest indexed $\bar{c}_i$ such that

$A_{5t+1}(s) = A_{5t}(s)\hat{\ } f_t(s)$;
$A_{5t+2}(s) = A_{5t+1}(s)\hat{\ }\langle c_t\rangle$;
$A_{5t+3}(s) = A_{5t+2}(s)$;
$A_{5t+4}(s) = A_{5t+3}(s)\hat{\ } g_{i,j}(s)$, where $v(i,j) = t$; and
$A_{5t+5}(s) = A_{5t+4}(s)$;

and of course if there can be no such $\bar{c}_i$, then $A_{5t+1}(s)$ is undefined.

Choices during the construction will be influenced by a set of requirements $\{R_i \mid i < \omega\}$ having the natural priority ordering. Loosely speaking, the task associated with requirement R_{5n+i} will be, for

$i = 0$: to ensure that Γ_n is realized in $\mathcal{C}$;
$i = 1, 4$: to ensure that candidates associated with requirements of higher priority have their respective types amalgamated by a Γ_j;
$i = 2$: to ensure that $\langle c_0, \ldots, c_n\rangle$ realizes a Γ_j; and
$i = 3$: to ensure that $\mathcal{C}$ is homogeneous.

Specifically, we say that R_{5n+i} *is t-satisfied for* φ if

$i = 0$: $\Gamma_n(f_n(t)) \cup \{\chi_t, \varphi\}$ is consistent;
$i = 1$: $H_{3n}(t)(A_{5n+1}(t)) \cup \Gamma^t_n(f_n(t)) \cup H_{3n-1}(t)^t(A_{5n}(t)) \cup \{\chi_t, \varphi\}$ is consistent;
$i = 2$: $H_{3n+1}(t)(A_{5n+2}(t)) \cup H_{3n}(t)^t(A_{5n+1}(t)) \cup \{\chi_t, \varphi\}$ is consistent;
$i = 3$: If $v(k,j) = n$ and $\Gamma_j(\langle c_0, \ldots, c_k\rangle\hat{\ }\bar{x}) \cup \{\chi_t, \varphi\}$ is consistent, where Γ_j is a $k + v$-type for some $v > 0$, then $\Gamma_j(\langle c_0, \ldots, c_k\rangle\hat{\ } g_{k,j}(t)) \cup \{\chi_t, \varphi\}$ is consistent; *or* if either of the previous conditions fail, i.e. $\Gamma_j(\langle c_0, \ldots, c_k\rangle\hat{\ }\bar{x})$ is inconsistent or Γ_j is an m-type for some $m \leqslant k$, then $g_{k,j}(t) = \langle\ \rangle$;
$i = 4$: if $v(k,j) = n$ and $g_{k,j}(t) \neq \langle\ \rangle$, then

$$H_{3n+2}(t)\big(A_{5n+4}(t)\big) \cup H_{3n+1}(t)^t\big(A_{5n+2}(t)\big) \cup \Gamma^t_j\big(\langle c_0, \ldots, c_k\rangle\hat{\ } g_{k,j}(t)\big) \cup \{\chi_t, \varphi\}$$

is consistent; and if the condition fails, then the definition automatically holds.

Notice that to determine whether or not R_m is t-satisfied for ψ_i is a procedure uniformly effective in m, t, and i, as long as the construction is uniformly effective. Various requirements and functions will be *associated* in an obvious way: $R_{5n} - f_n$; $R_{5n+1} - f_n$, H_{3n}, and A_{5n+1}; $R_{5n+2} - H_{3n+1}$, A_{5n+2}, and A_{5n+3}; $R_{5n+3} - g_{i,j}$, where $v(i,j) = n$; and $R_{5n+4} - H_{3n+2}$, A_{5n+4}, and A_{5n+5}. In the construction that follows, if a function is defined on an argument t and has not previously been specified as undefined for $t+1$, then its value on $t+1$ is to be the same as its value on t. Also, requirements are either *active* or *inactive* at a particular stage.

The construction.

Stage 0. All functions are undefined at 0 (except A_0), all requirements are inactive, and $\theta_0 =_{\mathrm{df}} (c_0 = c_0)$.

Stage $t = 3s+1$. If $\theta_t\chi_t = \exists x\, \sigma(x)$, then for the least indexed c_i not occurring in $\theta_t =_{\mathrm{df}} \sigma(c_i)$; otherwise $\theta_t =_{\mathrm{df}} (c_0 = c_0)$.

Stage $t = 3s+2$. Fix the highest priority requirement R_{5n+i}, $i < 5$, that is inactive. This requirement is now active and

$i = 0$. If Γ_n is a k-type, then define $f_n(t)$ to be the least indexed $\bar{c}_m$ that is a k-tuple and no c_j occurring in $\bar{c}_m$ occurs in χ_t or ψ_s.

$i = 1$. If $A_{3n+i}(3s)$ is a k-tuple, then let $H_{3n}(t)$ be the least indexed Γ_m that is a k-type and such that

$$\Gamma_m\big(A_{3n+1}(3s)\big) \cup \Gamma_n^{3s}\big(f_n(3s)\big) \cup H_{3n-1}(3s)^{3s}\big(A_{3n}(3s)\big) \cup \{\chi_t\}$$

is consistent.

$i = 2$. If $H_{3n}(3s)$ is a k-type, then let $H_{3n+1}(t)$ be the least indexed Γ_m that is a $k+1$-type and such that

$$\Gamma_m\big(A_{5n+2}(3s)\big) \cup H_{3n}(3s)^{3s}\big(A_{5n+1}(3s)\big) \cup \{\chi_t\}$$

is consistent.

$i = 3$. Let $v^{-1}(n) = \langle k,j\rangle$. If Γ_j is an r-type $r \leqslant k+1$ or $\Gamma_j(\langle c_0,\ldots,c_k\rangle\hat{\ }\bar{x}) \cup \{\chi_t\}$ is inconsistent, then $g_{k,j}(t) =_{\mathrm{df}} \langle\ \rangle$. Otherwise define $g_{k,j}(t)$ as the least indexed $\bar{c}_m$ that is a $(p\text{-}k\text{-}1)$-tuple, where Γ_j is a p-tuple, and such that no c_u occurring in $\bar{c}_m$ occurs in χ_t or ψ_{3s}.

$i = 4$. If $v^{-1}(n) = \langle k,j\rangle$ and $g_{k,j}(3s) = \langle\ \rangle$, then $H_{3n+2}(t) = H_{3n+2}(3s+2)$; otherwise, if $A_{5n+4}(3s)$ is an r-tuple, then let $H_{3n+2}(t)$ be the least indexed Γ_m that is an r-type and such that

$$\Gamma_m\big(A_{5n+4}(3s)\big) \cup H_{3n+1}(3s)^{3s}\big(a_{5n+2}(3s)\big) \cup \Gamma_j^{3s}\big(\langle c_0,\ldots,c_k\rangle\hat{\ }g_{k,j}(3s)\big) \cup \{\chi_t\}$$

is consistent.

Regardless of the value of i, $\theta_t =_{\mathrm{df}} (c_0 = c_0)$.

Stage $t = 3s+3$. I. There is an active R_{5n+2} such that R_i is t-satisfied for ψ_s^k, $k = 0, 1$, $i < 5n+2$; R_{5m+1} is *not* t-satisfied for $\neg((\chi_t \wedge \psi_s^k)^*[A_{5m+1}(3s)])$ for either $k=0$ or $k=1$ (neither value of k produces t-satisfaction), $m < n$; and R_{5n+1} is t-satisfied for $\neg((\chi_t \wedge \psi_s^k)^*[A_{5n+2}(3s)])$ for at least one of $k = 0, 1$. Fix the greatest such n. If $A_{5n+2}(3s)$ is an r-tuple, then let Σ_m be the least indexed Σ_j that is an r-type and such that $\Sigma_m(A_{5n+2}(3s)) \cup \{\chi_t\}$ is consistent. [R_{5n+2} will be referred to as the controlling requirement.]

A. $\Sigma_m(A_{5n+2}(3s)) \cup \{\chi_t, \psi_s^k\}$ is *inconsistent* for some $k = 0, 1$ (actually at most one). For the least such k, $\theta_t =_{\mathrm{df}} \psi_s^k$.

B. Otherwise. Then $\theta_t =_{\mathrm{df}} (c_0 = c_0)$.

II. Otherwise. Let m_k be the least i such that R_i is not t-satisfied for ψ_s^k, or $m_k = t$ if no such $i \leqslant t$ exists, $k = 0, 1$. Now let $k = 0$ if $m_0 \geqslant m_1$, $k = 1$ otherwise, and $\theta_t =_{\mathrm{df}} \psi_s^k$.

After θ_t has been defined let R_{5m+i} be the requirement of highest priority that is not $(t + 1)$-satisfied for $(c_0 = c_0)$. Then R_j is now inactive and all associated functions are undefined for $j \geqslant 5m + i$, and if $i = 1$ or $i = 4$, then also for $j = 5m + i - 1$.

This ends the construction.

LEMMA 3. *The construction is uniformly effective.*

PROOF. The details of this will be left to the reader, but we note that all types considered are (uniformly) recursive and that if $\varphi(\bar{x}) \in L$ is a formula consistent with $\mathrm{Th}(\mathfrak{A})$, then there is always an $i < \omega$ such that $\varphi(\bar{x}) \in \Gamma_i(\bar{x})$.

LEMMA 4. $\{\theta_i \mid i < \omega\} \cup \mathrm{Th}(\mathfrak{A})$ *is consistent.*

PROOF. This is also easy to check, since in fact θ_t is always specified so that $\Gamma_0(A_1^*) \cup \{\chi_{t+1}\}$ is consistent.

LEMMA 5. f_i^*, $g_{i,j}^*$, H_i^*, A_i^* *all exist and* $\{\theta_i \mid i < \omega\}_{\restriction L \cup \mathrm{rg}\, A_i^*}$ *is complete*, $i, j < \omega$.

PROOF. The proof is by induction on the index of the associated R_i. Simultaneously we will prove that for all $n < \omega$ there is a $t < \omega$ such that for all

$$\varphi \in \{\psi_i \mid i < \omega\}_{\restriction L \cup \mathrm{rg}\, A_{5n+2}^*} \quad H_{3n}^*(A_{5n+1}^*) \vdash [\chi_t \to \varphi^k] \quad \text{for some } k = 0, 1.$$

Since R_0 has the highest priority it is easy to see from the construction that $f_0^* = f_0(1)$ and $H_0^* = 1$. Now, for each $\psi_j \in L \cup \mathrm{rg}\, f_0^*$ R_0 is $(t + 1)$-satisfied for ψ_j^k for exactly one value of $k = 0, 1$. Since R_0 has the highest priority, it follows from the instructions at stages $3s + 3$ that for that value of k, $\psi_j^k \in \{\theta_i \mid i < \omega\}$. This proves the lemma for R_0 and R_1. So assume that the lemma is true for R_i, $i < 5n$. Thus H_{3n-1}^*, A_{3n}^* exist and $\{\theta_i \mid i < \omega\}_{\restriction L \cup \mathrm{rg}\, A_{3n}^*}$ is complete. Thus from the construction it is obvious that $\{\theta_i \mid i < \omega\}_{\restriction L \cup \mathrm{rg}\, A_{3n}^*}$ is in fact $H_{3n-1}^*(A_{3n}^*)$. Since H_{3n-1}^* and Γ_n are both types that are realized in $\mathfrak{C}$, there is a Γ_k such that $H_{3n-1}^*(\bar{x}) \cup \Gamma_n(\bar{y}) \subset \Gamma_k(\bar{x}, \bar{y})$. (By the choice made in the case $i = 0$ of stages $3s + 2$ for $f_n(3s + 2)$, and the construction, it is not necessary to invoke homogeneity in order to justify what follows.) Fix the least such $k = k_1$. Let t_0 be a stage after which every R_i is always active, $i < 5n$. Let $t_1 > t_0$ be a stage such that for each $i < k_1$, $H_{3n-1}^{*t_1}(\bar{x}) \cup \Gamma_n^{t_1}(\bar{y}) \not\subset \Gamma_i(\bar{x}, \bar{y})$. Then it is easy to see by the above remark, the construction, and the choice of t_0, t_1 that for all $t > t_1 + 2$ $H_{3n}^* = \Gamma_{k_1}$, $f_n(t) = f_n(t + 1)$ and $A_{5n+1}(t) = A_{5n+1}(t + 1)$. This completes R_{5n} and R_{5n+1} except to show that $\{\theta_i \mid i < \omega\}_{\restriction L \cup \mathrm{rg}\, A_{5n+1}^*}$ is complete. By the induction hypotheses fix $t_2 > t_1$ such that for all $m < n$

$$(*) \quad \begin{array}{l} \text{for all } \varphi \in \{\psi_i \mid i < \omega\}_{\restriction L \cup \mathrm{rg}\, A_{5m+2}^*} \text{ there is a } k = 0, 1 \text{ such} \\ \text{that } H_{3m}^*(A_{5m+1}^*) \vdash [\chi_{t_2} \to \varphi^k]. \end{array}$$

Next fix $\psi_s \in L \cup \operatorname{rg} A^*_{5n+1}$ in order to show that $\psi_s^k \in \{\theta_i \mid i < \omega\}$ for some $k = 0, 1$. By the assumption on the enumeration $\{\psi_i \mid i < \omega\}$, we may assume without loss that $s > t_2$. Now consider stage $3s + 3$. It is enough to show that the defining case is II, since then θ_{3s+3} is ψ_s^k for one of $k = 0, 1$. Suppose first that for some $m < n$ R_{5m+2} is the controlling condition in case I. Then for some value of $k = 0, 1$ $\neg((\chi_{3s+3} \hat{\ } \psi_s^k)^*[A^*_{5m+2}])$ does $(3s + 3)$-satisfy R_{5m+1}. Thus by the choice of t_2 and $(*)$ it follows that $H^*_{5m}(A^*_{5m+1}) \vdash \chi_{t_2} \to \neg((\chi_{3s+2} \wedge \psi_s^k)^*[A^*_{5m+2}])$ for that value of k. But then it is easy to see that R_{5m+1} is not $(3s + 3)$-satisfied for ψ_s^k, which contradicts the instructions in I. On the other hand ψ_s^k is in $L \cup \operatorname{rg} A^*_{5n+1}$ and so *no* R_{5m+2} for $m > n$ can be the controlling condition either, since R_{5n+1} can only be $(3s + 3)$-satisfied for ψ_s^k for at most one value of k. Thus the defining case is II.

In order to prove the lemma for R_{5n+2}, note first that $A^*_{5n+2} = A^*_{5n+1} \hat{\ } \langle c_n \rangle$; so fix $t_3 > t_2$ such that for all $t \geqslant t_3$ $A_{5n+1}(t) = A^*_{5n+1}$. The completeness of $\{\theta_i \mid i < \omega\}_{\restriction L \cup \operatorname{rg} A^*_{5n+2}}$ is the first claim to be established. Therefore choose any $\psi_s \in L \cup \operatorname{rg} A^*_{5n+2}$ in order to show that $\psi_s^k \in \{\theta_i \mid i < \omega\}$ for one of $k = 0, 1$. Again we may assume that $s > t_3$ and just as in the last paragraph the only difficulty possible is that the defining condition at stage $3s + 3$ is I.B. However, the argument is identical to the one in the previous paragraph for eliminating the possibility that R_{5m+2} is the controlling requirement for $m \neq n$. Thus the only new alternative is that R_{5n+2} might be the controlling requirement. But then since $\psi_s \in L \cup \operatorname{rg} A^*_{5n+2}$, ψ_s^k can be consistent with the appropriate $\Sigma_m(A^*_{5n+2})$ in I. for at most one value of k. Thus the defining condition would be I.A. and so the claim is proven.

We next establish the additional inductive assumption that was asserted, i.e. that $(*)$ holds for $m = n$. Assume that it fails in order to obtain a contradiction; thus for every $t < \omega$ there is a $\psi_s \in L \cup \operatorname{rg} A^*_{5n+2}$ such that

$$(\dagger) \qquad H^*_{3n}(A^*_{5n+1}) \nvdash (\chi_t \to \psi_s^k), \qquad k = 0, 1.$$

Now by Lemma 1 and what has just been established, $\{\theta_i \mid i < \omega\}_{\restriction L \cup \operatorname{rg} A^*_{5n+2}}$ is a recursive type of $\operatorname{Th}(\mathfrak{A})$ and so is a Σ_i for some $i < \omega$. Let m be the least such i. By the choice of m fix $t_4 > t_3$ such that either Σ_i is not a k-type, where A^*_{5n+2} is a k-tuple, or $\Sigma_i(A^*_{5n+2}) \cup \{\chi_{t_4}\}$ is inconsistent, $i < m$. By our assumption, there is a $\psi_s \in L \cup \operatorname{rg} A^*_{5n+2}$ satisfying $H^*_{3n}(A^*_{5n+1}) \nvdash (\chi_{t_4} \to \psi_s^k)$, $k = 0, 1$. Fix the least indexed such ψ_s satisfying $3s + 3 > t_4$. Assume first that there is a least j, $t_4 \leqslant j < 3s + 3$ such that $H^*_{3n}(A^*_{5n+1}) \vdash [(\chi_j \wedge \theta_j) \to \psi_s^k]$ for some $k = 0, 1$. Thus $H^*_{3n}(A^*_{5n+1}) \nvdash [\chi_j \to \psi_s^k]$, $k = 0, 1$. From the construction it follows that $j = 3r + 3$ for some $r < s$ and that the defining condition at stage j is not I.B. Since $\psi_s \in L \cup \operatorname{rg} A^*_{5n+2}$, it also follows by the choice of j that

$$H^*_{3n}(A^*_{5n+1}) \nvdash \left[\chi_j \to (\chi_j \wedge \theta_j)^*[A^*_{5n+2}]\right].$$

Thus R_{5n+1} is t-satisfied for $\neg((\chi_j \wedge \theta_j)^*[A^*_{5n+2}])$. In fact, the claim is now that R_{5n+1} is the controlling condition for stage j. To see this note that by $(\dagger)$ and the induction hypothesis $(*)$ that R_{5u+1} is j-satisfied for ψ_s^k and is not j-satisfied for

$$\neg\left((\chi_j \wedge \psi_s^k)^*[A^*_{5u+2}]\right), \qquad k = 0, 1, u < n.$$

By the instructions in I of the construction it follows that R_{5n+2} is the controlling requirement. However, since in the case considered we are not in defining condition

I.B., it follows that θ_i is specified so that χ_{j+1} is inconsistent with $\Sigma_m(A^*_{5n+2})$ (since $j \geqslant t_4$). This contradicts the choice of Σ_m. So we are reduced to the case where $H^*_{3n}(A^*_{5n+1}) \nvdash (\chi_{3s+3} \to \psi_s^k)$, $k = 0, 1$. But again the same argument as before shows that the controlling requirement at stage $3s + 3$ would then be R_{5n+2}. Of course the guarantee that the defining condition is not I.B. is not the same, but there still is one. This is simply because $\psi_s \in L \cup \operatorname{rg} A^*_{5n+2}$ and so ψ_s^k can be consistent with $\Sigma_m(A^*_{5n+2})$ for at most one value of k. Therefore θ_{3s+3} is determined so that χ_{3s+4} is inconsistent with $\Sigma_m(A^*_{5n+2})$, which is again a contradiction. This establishes $(*)$ for $m = n$.

So let $t_4 > t_3$ be large enough that $(*)$ holds at stage t_4. Now there must be some Γ_i such that $H^*_{3n}(A^*_{5n+1}) \cup \Gamma_i(A^*_{5n+2}) \cup \{\chi_{t_4+1}\}$ is consistent and Γ_i is a k-type (as before). Fix the least such i to be r. Then it is easy to see that $H^*_{3n+1} = \Gamma_r$, since R_{5s+2} can never be the controlling requirement for $s \leqslant n$ after stage t_4.

So now fix $t_5 > t_4$ such that for all $t > t_5$, $H^*_{3n+1} = H_{3n+1}(t)$ as we move on to R_{5n+3} and R_{5n+4}. Let $v^{-1}(n) = \langle k_0, j_0\rangle$. The choice of v ensures that

$$\{\theta_i \mid i < \omega\}_{\restriction L \cup \{c_i \mid i \leqslant k_0\}} \subset \{\theta_i \mid i < \omega\}_{\restriction L \cup \operatorname{rg} A^*_{5n+2}}$$

and is thus a complete type Γ_{i_0} for some i_0. If $\Gamma_{i_0}(c_0, \ldots, c_{k_0}) \not\subset \Gamma_{j_0}(\langle c_0, \ldots, c_{k_0}\rangle \wedge \bar{x})$, then it is easy to see that $g^*_{k_0, j_0} = \langle\ \rangle$ and $H^*_{3n+2} = H^*_{3n+1}$. If it is a subset, then since $\mathcal{C}$ is homogeneous, there is an r such that

$$\Gamma_{j_0}\big(\langle c_0, \ldots, c_{k_0}\rangle \wedge \bar{x}\big) \cup H^*_{3n+1}(A^*_{5n+2}) \subset \Gamma_r(A^*_{5n+2} \wedge \bar{x}).$$

For the least such r there is at least $t_6 > t_5$ such $H^*_{3n+2} = \Gamma_r$ and $A^*_{5n+4} = A_{5n+4}(t_6)$. These claims as well as the completeness of $\{\theta_i \mid i < \omega\}_{\restriction L \cup \operatorname{rg} A^*_{5n+4}}$ are similar to previous ones in their proof and we leave them to the reader. This completes the lemma.

LEMMA 6. *$\{\theta_i \mid i < \omega\}$ is complete and recursive.*

PROOF. Lemmas 3 and 5.

LEMMA 7. *$\{\theta_i \mid i < \omega\}$ is the complete diagram of a decidable model $\mathcal{C}$.*

PROOF. Lemmas 4 and 6 and the Henkin construction.

LEMMA 8. *Every Γ_i is realized in $\mathcal{C}$, $i < \omega$.*

PROOF. This follows from Lemma 5 since in fact Γ_i is realized in $\mathcal{C}$ by the equivalence class of f_i^*.

LEMMA 9. *Only the types in $\{\Gamma_i \mid i < \omega\}$ are realized in $\mathcal{C}$.*

PROOF. This again follows from Lemma 5 since by that lemma H_i^* exists for all $i < \omega$. In particular, H^*_{3n+1} exists for all $n < \omega$.

LEMMA 10. *$\mathcal{C}$ is homogeneous.*

PROOF. Suppose $\Gamma_r(\bar{x}) \subset \Gamma_j(\bar{x}, \bar{y})$ and $\langle c_{i_0}, \ldots, c_{i_r}\rangle$ realizes Γ_r in $\mathcal{C}$. Then fix a j_0 such that $\Gamma_j(\langle x_{i_0}, x_{i_1}, \ldots, x_{i_r}\rangle \wedge \bar{y}) \subset \Gamma_{j_0}(\langle x_0, x_1, \ldots, x_{i_r}\rangle \wedge \bar{y})$ and $\langle c_0, \ldots, c_{i_r}\rangle$ realizes $\Gamma_{j_0}(\bar{x}, \bar{y})_{\restriction \bar{x}}$. By Lemma 5 $g^*_{i_r, j_0}$ exists, and by the construction it follows that

$\langle c_0,\ldots,c_{i_r}\rangle\hat{}g^*_{i_r,j_0}$ realizes Γ_{j_0} in $\mathcal{C}$, and thus $\langle c_{i_0},\ldots,c_{i_r}\rangle\hat{}g^*_{i_r,j_0}$ realizes Γ_j in $\mathcal{C}$. This completes the proof of the theorem.

COROLLARY 3. *Assume that T has a decidable recursively saturated model. Then for every homogeneous model $\mathcal{Q}$ of T, $\mathcal{Q}$ is decidable iff the set of type realized by $\mathcal{Q}$ is a Σ^0_2 set of recursive types.*

PROOF. The recursively saturated decidable model provides an effective enumeration of all recursive types of T.

BIBLIOGRAPHY

1. L. Harrington, *Recursively presentable prime models*, J. Symbolic Logic **392** (1974), 305–309.
2. T. S. Millar, *Foundations of recursive model theory*, Ann. Math. Logic **13** (1978), 45–72.
3. C. C. Chang and H. J. Keisler, *Model theory*, American Elsevier, New York, 1973.
4. H. Rogers, Jr., *Theory of recursive functions and effective compatibility*, McGraw-Hill, New York, 1967.
5. T. S. Millar, *Homogeneous models and decidability*, Pacific J. Math. **91** (1980), 407–418.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706